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Role of viscosity stratification in the stability of two-layer flow down an incline

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The stability of the flow of two layers of viscous liquids down an incline is investigated. The problem is governed by two competing long-wavelength modes associated with the surfaces. One mode, calling it the first mode, is faster than the second one. When the two layers have the same coefficient of dynamic viscosity, the second mode was found earlier by the author to be the governing mode for most cases. The situation is greatly altered when viscosity is not the same for the two layers. The second mode is dramatically stabilized for the range of viscosity ratio, m, less than unity, and the first mode is now generally the governing mode in that range. The overall effect is stabilizing compared with $m = 1$.

A relative stability index is also introduced to compare the result with that of the homogeneous case. It is found that the presence of the upper layer is generally destabilizing compared with that of a homogeneous fluid of the same total depth.

1. Introduction

The stability of two-layered viscous flow down an incline was investigated by the author (1965*a*, *b*, hereinafter referred to as I and II). Attention was focused on the effect of density variation by assuming the same viscosity for the two layers. It was found that there existed two modes of long-wave instabilities with the second mode generally governing, often causing instability to set in at any Reynolds number however small. While there are some important situations, such as fresh water and salt water, in which the coefficient of dynamic viscosity is essentially the same for the two layers, this is usually not the case for two physically distinct liquids. The effect of this variation of viscosity on the two modes of instability is therefore of definite interest in physically realistic cases. It is therefore thought worthwhile to investigate the problem taking into account the effect of viscosity difference, especially in view of the fact that Yih (1967) has shown the destabilizing effect of viscosity stratification in plane Poiseuille and Couette flows. It will be seen in this paper that in the present case viscosity stratification is actually strongly stabilizing on the second mode although it could be mildly destabilizing on the first mode. Indeed, the overall effect will be seen to be stabilizing, and, depending on the values of the ratios of density, viscosity, and depth, the two modes compete to govern the stability of the system.

The linearized stability problem is set up in the usual manner. The resulting two Orr–Sommerfeld equations for the two layers, plus interfacial, boundary, and free surface conditions, form an eigenvalue problem. The ratios of density, depth, and viscosity of the two layers appear as extra parameters of the problem. A method of long-wave approximation introduced by Yih (1963) is used to solve the problem.

2. Formulation of the stability problem

A brief derivation of the basic flow and perturbation equations will now be given together with a more detailed consideration of the boundary conditions and the eigenvalue problem.

The basic flow

On assuming parallel flow and using the X–Y co-ordinate axes as shown in figure 1, the solutions to the Navier–Stokes equations of the two-layered flow are, in non-dimensional form,

$$U_1 = a_1 y^2 + b_1 y + k_1 \quad (-\delta \leqslant y \leqslant 0), \tag{1}$$

$$U_2 = a_2 y^2 + b_2 y + k_2 \quad (0 \leqslant y \leqslant 1), \tag{2}$$

in which $a_1 = -\dfrac{r}{m}\dfrac{K}{2}, \quad b_1 = \dfrac{r\delta K}{m}, \quad k_1 = \left(\dfrac{K}{2} + r\delta K\right),$

$a_2 = -\tfrac{1}{2}K, \quad b_2 = -r\delta K \quad \text{and} \quad k_2 = k_1 = (\tfrac{1}{2}K + r\delta K).$

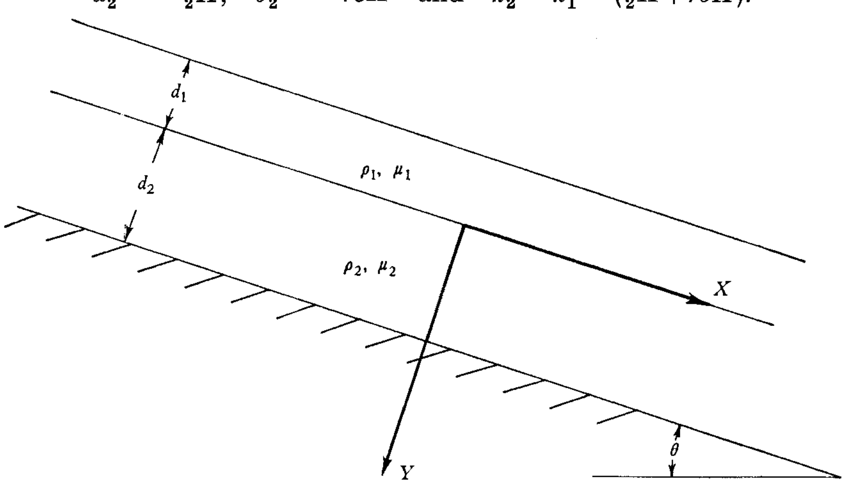

FIGURE 1. Definition sketch.

The symbols above are defined as follows: $\delta = d_1/d_2$ is the ratio of depths, $r = \rho_1/\rho_2$ is the ratio of densities, $m = \mu_1/\mu_2$ is the ratio of viscosities, and

$$K \equiv (1+\delta)[r(\tfrac{1}{2}\delta + \delta^2 + \delta^3/3m) + (\tfrac{1}{3} + \tfrac{1}{2}\delta)]^{-1}.$$

The reference length is d_2 and the reference velocity is the average velocity $\overline{U}_a$ given by

$$\overline{U}_a = \frac{\rho_2 g \sin\theta d_2^2}{\mu_2 K}. \tag{3}$$

The subscript 1 denotes the upper fluid while 2 denotes the lower fluid. The

Reynolds numbers and Froude number are now defined to be

$$R_1 = \frac{\rho_1 \overline{U}_a d_2}{\mu_1}, \quad R_2 = \frac{\rho_2 \overline{U}_a d_2}{\mu_2}, \quad F = \overline{U}_a/(gd_2)^{\frac{1}{2}}.$$

It then follows that $R_1 = (r/m) R_2$ and $KF^2 = R_2 \sin\theta$. In deriving the mean flow velocities U_1 and U_2 we have made use of the no-slip condition at the solid boundary and the interface together with the conditions of zero shear at the free surface and equal shear at the interface. We also note that, in the y-direction, the mean flow pressure is hydrostatic,

$$\frac{dP_1}{dy} = \frac{\cos\theta}{F^2} \quad \text{and} \quad \frac{dP_2}{dy} = \frac{\cos\theta}{F^2},$$

where P_1 and P_2 are the mean flow pressures in the upper and lower layers normalized by $\rho_1 \overline{U}_a^2$ and $\rho_2 \overline{U}_a^2$ respectively. Thus, at the interface $y = 0$, the equality of pressure yields $rP_1 = P_2$.

Perturbation equations

Following the usual procedure, the equations governing the disturbance motion in the two layers are derived from the Navier–Stokes equations on introducing infinitesimal two-dimensional disturbances to the basic flow. On decomposing the non-dimensional disturbance streamfunctions ψ_1 and ψ_2 and the disturbance pressures into normal modes, there results, upon elimination of the pressures, the Orr–Sommerfeld equations:

$$\phi_1^{\text{iv}} - 2\alpha^2\phi_1'' + \alpha^4\phi_1 = i\alpha R_1\{(U_1 - c)(\phi_1'' - \alpha^2\phi_1) - U_1''\phi_1\}, \tag{4}$$

in $-\delta \leqslant y \leqslant 0$, where the superscripts denote differentiation with respect to y, and

$$\phi_2^{\text{iv}} - 2\alpha^2\phi_2'' + \alpha^4\phi_2 = i\alpha R_2\{(U_2 - c)(\phi_2'' - \alpha^2\phi_2) - U_2''\phi_2\}, \tag{5}$$

in $0 \leqslant y \leqslant 1$. In the above, ϕ_1 and ϕ_2 are defined through the relations:

$$\psi_1 = \phi_1(y)\exp[i\alpha(x - ct)] \quad \text{and} \quad \psi_2 = \phi_2(y)\exp[i\alpha(x - ct)], \tag{6}$$

in which α is the dimensionless wave-number defined by $2\pi d_2/\lambda$, λ being the wavelength, and $c = c_r + ic_i$ is the dimensionless wave velocity normalized by $\overline{U}_a$.

Boundary conditions

We first give the kinematic conditions at the interface and free surface. Let the equation of the free surface be given by $y = -\delta + \xi(x,t)$, and the interface by $y = \eta(x,t)$. The linearized kinematic conditions are then

$$\frac{\partial \xi}{\partial t} + U_1 \frac{\partial \xi}{\partial x} = -\frac{\partial \psi_1}{\partial x}$$

at the free surface and $\quad \dfrac{\partial \eta}{\partial t} + U_2 \dfrac{\partial \eta}{\partial x} = -\dfrac{\partial \psi_2}{\partial x} = -\dfrac{\partial \psi_1}{\partial x}$

at the interface. It then follows that

$$\xi = [\phi_1(-\delta)/c_1] \exp [i\alpha(x-ct)], \tag{7}$$

and

$$\eta = [\phi_2(0)/c_2] \exp [i\alpha(x-ct)], \tag{8}$$

where $c_1 \equiv c - U_1(-\delta)$ and $c_2 \equiv c - U_2(0)$.

We now formulate the boundary conditions. We denote the total non-dimensional velocity and pressure by (u, v) and p with $\overline{U}_a$ as the reference velocity and $\rho \overline{U}_a^2$ as the normalizing quantity for pressure. The subscripts 1 and 2 will be used to designate the upper and lower layers respectively.

At the free surface the shear stress must vanish, and the normal stress must balance the normal stress induced by surface tension. Thus we have

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = 0$$

and

$$\left(-p_1 + \frac{2}{R_1}\frac{\partial v_1}{\partial y}\right) + S_1 \frac{\partial^2 \xi}{\partial x^2} = 0,$$

where $S_1 \equiv T_1/(\rho_1 \overline{U}_a^2 d_2)$, T_1 being the surface tension. At the interface, the total velocity components must be continuous; i.e. $u_1 = u_2$ and $v_1 = v_2$. Also, the shear must be continuous,

$$m\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\right) = \frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y}.$$

The difference of the normal stresses must be balanced by the normal stress induced by the interfacial surface tension:

$$\left(-p_2 + \frac{2}{R_2}\frac{\partial v_2}{\partial y}\right) - \left(-p_1 + \frac{2}{R_1}\frac{\partial v_1}{\partial y}\right) r + S_2 \frac{\partial^2 \eta}{\partial x^2} = 0,$$

where $S_2 \equiv T_2/(\rho_2 \overline{U}_a^2 d_2)$, T_2 being the interfacial surface tension. Lastly, at the solid boundary $y = 1$, we have $u_2 = 0$ and $v_2 = 0$.

To the first order in the disturbance quantities, the above eight boundary conditions may be written as

(i) $\phi_1''(-\delta) + (\alpha^2 + U_1''(-\delta)/c_1)\phi_1(-\delta) = 0,$

(ii) $[\alpha\{(r/m)K \cot\theta + \alpha^2 S_1 R_1\}/c_1]\phi_1(-\delta) + \alpha(R_1 c_1 + 3i\alpha)\phi_1'(-\delta) - i\phi_1'''(-\delta) = 0,$

(iii) $\phi_1(0) = \phi_2(0),$

(iv) $\phi_1'(0) - \phi_2'(0) = \dfrac{\phi_1(0)}{c_2}[U_2'(0) - U_1'(0)],$

(v) $m\phi_1''(0) - \phi_2''(0) + [mU_1''(0) - U_2''(0)]\dfrac{\phi_2(0)}{c_2} + \alpha^2(m-1)\phi_2(0) = 0,$

(vi) $c_2[\phi_2'''(0) - m\phi_1'''(0)] + i\alpha R_2[c_2^2(\phi_2'(0) - r\phi_1'(0))$
$+ (U_2'(0) - rU_1'(0))c_2\phi_2(0)] - 3c_2\alpha^2[\phi_2'(0) - m\phi_1'(0)]$
$+ i\alpha[K(1-r)\cot\theta + \alpha^2 S_2 R_2]\phi_2(0) = 0,$

(vii) $\phi_2(1) = 0,$

(viii) $\phi_2'(1) = 0.$

To obtain the above set of equations the mean flow quantities should be evaluated at $y = -\delta+\xi$ for the free surface conditions and at $y = \eta$ for the interface conditions. However, since ξ and η are perturbation quantities which are small, we need only take the leading terms, consistent with previous linearization, of the Taylor series expansions of quantities of interest and evaluate them at $y = -\delta$ and $y = 0$.

Eigenvalue problem

Equations (4) and (5) together with boundary conditions (i) to (viii) form the eigenvalue problem we wish to solve with c as the eigenvalue. The general solutions of (4) and (5) will contain eight arbitrary constants. Substitution of these solutions into the eight homogeneous boundary conditions will yield eight homogeneous algebraic equations for the eight constants. The vanishing of the determinant of the coefficients will give the secular equation in the form $c = c(\alpha, R_2, r, m, \delta, \theta)$. Since c is complex, this relationship can be resolved into $c_r = c_r(\alpha, R_2, r, m, \delta, \theta)$ and $c_i = c_i(\alpha, R_2, r, m, \delta, \theta)$. Putting $c_i = 0$, the equation $c_i(\alpha, R_2, r, m, \delta, \theta) = 0$ represents the relationship between α and R_2 for given values of r, m, δ and θ.

3. Solution for long waves

It will be seen that all of the relevant information on the problem can be obtained from examining long-wavelength disturbances. We therefore adopt the method used by Yih (1963) and introduce perturbation series of ϕ_1, ϕ_2 and c in the form

$$\left.\begin{aligned} \phi_1 &= \phi_{10}+\alpha\phi_{11}+\alpha^2\phi_{12}+\ldots,\\ \phi_2 &= \phi_{20}+\alpha\phi_{21}+\alpha^2\phi_{22}+\ldots,\\ c &= c_0+\alpha\Delta c+\ldots. \end{aligned}\right\} \tag{9}$$

Substitution of (9) into (4) and (5) and (i) to (viii), and collecting terms to the zeroth order in α, yield

$$\phi_{10}^{\mathrm{iv}} = 0 \quad (-\delta \leqslant y \leqslant 0), \tag{10}$$

$$\phi_{20}^{\mathrm{iv}} = 0 \quad (0 \leqslant y \leqslant 1), \tag{11}$$

$$\phi_{10}''(-\delta)+\frac{2a_1}{c_{10}}\phi_{10}(-\delta) = 0, \quad \phi_{10}'''(-\delta) = 0, \quad \phi_{10}(0)-\phi_{20}(0) = 0,$$

$$\phi_{10}'(0)-\phi_{20}'(0)-(b_2-b_1)\frac{\phi_{10}(0)}{c_{20}} = 0, \quad m\phi_{10}''(0)-\phi_{20}''(0)+2(ma_1-a_2)\frac{\phi_{20}(0)}{c_{20}} = 0,$$

$$\phi_{20}'''(0)-m\phi_{10}'''(0) = 0, \quad \phi_{20}(1) = 0, \quad \phi_{20}'(1) = 0,$$

where $c_{10} = c_0-U_1(-\delta)$ and $c_{20} = c_0-U_2(0)$. The solution is straightforward. After some calculation, we find

$$\begin{aligned} c_{20} = {} & -\tfrac{1}{2}(a_2+a_1\delta^2+2a_1m\delta+l) \\ & \pm [(a_2+a_1\delta^2+2a_1m\delta+l)^2+4[(ma_1-a_2)(l+a_1\delta^2)+a_1m\delta(b_2-b_1)]]^{\frac{1}{2}}, \end{aligned} \tag{12}$$

where $l = U_2(0)-U_1(-\delta)$. The plus and minus signs in front of the radical correspond to two different modes. We shall call the mode associated with the plus sign the first mode and the minus sign the second mode. The eigenfunctions are

$$\phi_{10}(y) = 1+B_1 y+D_1 y^2, \tag{13}$$

$$\phi_{20}(y) = 1-2y+y^2, \tag{14}$$

where $\quad B_1 = (b_2-b_1-2c_{20})/c_{20}\quad$ and $\quad D_1 = -\left[\dfrac{ma_1-a_2}{c_{20}}-1\right]/m.$

As pointed out in I, the oscillations of the two surfaces are in phase for the first mode while they are mainly 180° out of phase for the second mode. The wave velocity c_0 for the two modes is shown in figures 2 and 3 as a function of m for various values of δ and $r = 0{\cdot}9$. It may be noted that the second mode is slower than the first one. To the first order in α, we have

$$\phi_{11}^{\text{iv}} = iR_1\{(U_1-c_0)\phi_{10}''-U_1''\phi_{10}\} \quad (-\delta \leqslant y \leqslant 0), \tag{15}$$

$$\phi_{21}^{\text{iv}} = iR_2\{(U_2-c_0)\phi_{20}''-U_2''\phi_{20}\} \quad (0 \leqslant y \leqslant 1), \tag{16}$$

and the boundary conditions are

$$\phi_{11}''(-\delta)+\frac{2a_1}{c_{10}}\phi_{11}(-\delta) = 2a_1\frac{\Delta c}{c_{10}^2}\phi_{10}(-\delta),$$

$$\phi_{11}'''(-\delta) = -i\{[\{(r/m)K\cot\theta+\alpha^2 S_1 R_1\}/c_{10}]\phi_{10}(-\delta)+R_1 c_{10}\phi_{10}'(-\delta)\},$$

$$\phi_{11}(0)-\phi_{21}(0) = 0,$$

$$\phi_{11}'(0)-\phi_{21}'(0)-\frac{\phi_{11}(0)}{c_{20}}(b_2-b_1) = -\frac{\phi_{10}(0)\,\Delta c}{c_{20}^2}(b_2-b_1),$$

$$m\phi_{11}''(0)-\phi_{21}''(0)+2(ma_1-a_2)\frac{\phi_{21}(0)}{c_{20}} = 2(ma_1-a_2)\frac{\phi_{20}(0)}{c_{20}^2}\Delta c,$$

$$\begin{aligned} c_{20}[\phi_{21}'''(0)-m\phi_{11}'''(0)] = {} & -iR_2[c_{20}^2(\phi_{20}'(0)-r\phi_{10}'(0))+(b_2-rb_1)c_{20}\phi_{20}(0)] \\ & -i[K(1-r)\cot\theta+\alpha^2 S_2 R_2]\phi_{20}(0), \end{aligned}$$

$$\phi_{21}(1) = 0,$$

$$\phi_{12}'(1) = 0.$$

After some rather lengthy calculations, we obtain

$$\Delta c = i\{(G/H)R_2-[(\Phi/H)\cot\theta+(\Theta R_2 S_2/H)\alpha^2]\}, \tag{17}$$

where

$$H = \frac{2}{m}\left(1+\frac{a_1\delta^2}{c_{10}}\right)(ma_1-a_2)\frac{1}{c_{20}^2}-\frac{2a_1}{c_{10}^2}(1-B_1\delta+D_1\delta^2)+\frac{2a_1\delta}{c_{10}}\frac{(b_2-b_1)}{c_{20}^2},$$

$$\begin{aligned} G = {} & -\frac{r}{m}\left[K_1''(-\delta)+\frac{2a_1}{c_{10}}K_1(-\delta)+\frac{1}{6}\left(\frac{2a_1\delta^3}{c_{10}}+6\delta\right)K_1'''(-\delta)\right. \\ & +\frac{c_{10}}{6}(B_1-2D_1\delta)\left(\frac{2a_1\delta^3}{c_{10}}+6\delta\right)+\frac{m}{3}\frac{a_1\delta}{c_{10}}[c_{10}(B_1-2D_1\delta)+K_1'''(-\delta)] \\ & +\frac{2}{3}\left(1+\frac{a_1\delta^2}{c_{10}}\right)[c_{10}(B_1-2D_1\delta)+K_1'''(-\delta)]-\frac{a_1\delta}{c_{10}}\{\tfrac{1}{3}[c_{20}(-2-rB_1) \\ & +(b_2-rb_1)]-2[-2K_2(1)+K_2'(1)]\}-\frac{2}{m}\left(1+\frac{a_1\delta^2}{c_{10}}\right)\{\tfrac{1}{3}[c_{20}(-2-rB_1) \\ & +(b_2-rb_1)]-[-K_2(1)+K_2'(1)]\}, \end{aligned}$$

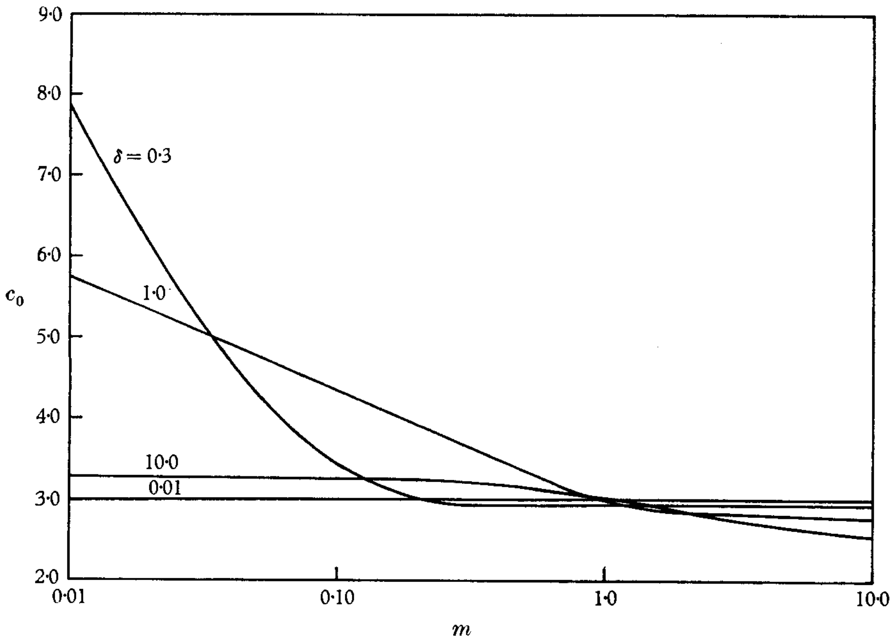

FIGURE 2. Wave velocity c_0 for first mode, with the ratio of density $r = \frac{9}{10}$.

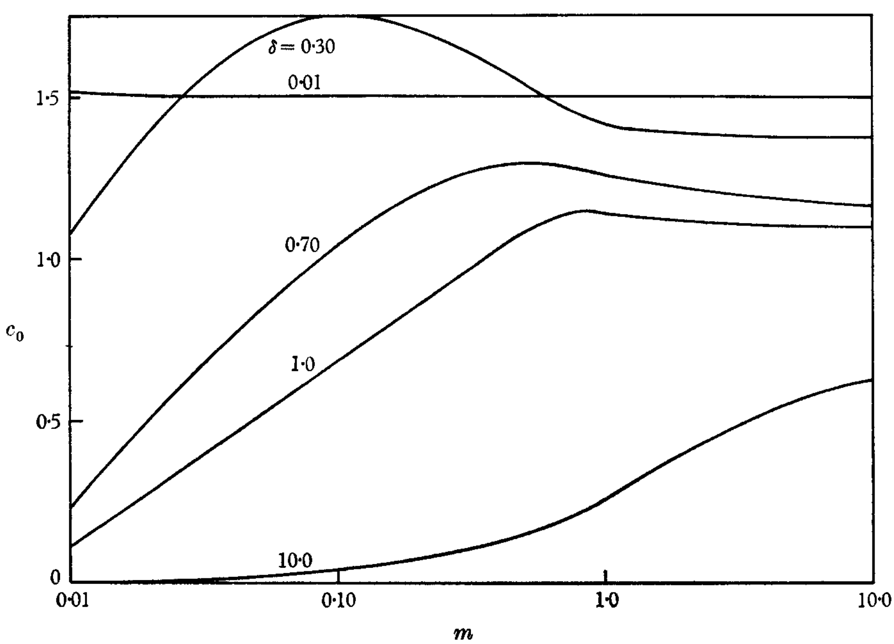

FIGURE 3. Wave velocity c_0 for second mode, with the ratio of density $r = \frac{9}{10}$.

$$\Phi = \frac{1}{6}\left(\frac{2a_1\delta^3}{c_{10}}+6\delta\right)\left(\frac{1-B_1\delta+D_1\delta^2}{c_{10}}\right)\frac{rK}{m}+\frac{1}{3}\frac{a_1\delta}{c_{10}}\left\{\frac{K(1-r)}{c_{20}}\right.$$

$$\left.+\frac{rK}{c_{10}}(1-B_1\delta+D_1\delta^2)\right\}+\frac{2}{3m}\left(1+\frac{a_1\delta^2}{c_{10}}\right)\left\{\frac{K(1-r)}{c_{20}}+\frac{rK}{c_{10}}(1-B_1\delta+D_1\delta^2)\right\},$$

$$\Theta = \frac{1}{3}\left\{qr\delta\left(\delta^2\frac{a_1}{c_{10}}+3\right)\phi_{10}(-\delta)/mc_{10}+\left(\frac{a_1\delta}{c_{10}}+\frac{2}{m}\left(1+\frac{a_1\delta^2}{c_{10}}\right)\right)\left[\frac{1}{c_{20}}+qr\phi_{10}(-\delta)/c_{10}\right]\right\},$$

$$K_1(y) = (b_1D_1-a_1B_1)y^5/60+[(k_1-c_0)D_1-a_1]y^4/12,$$

$$K_2(y) = (b_2+2a_2)y^5/60+(k_2-c_0-a_2)y^4/12,$$

$$q = S_1/S_2.$$

The calculations involved leading to these results have been independently checked by Mr C. Park. A crucial test of the correctness comes on putting $m = 1$, which reduces to the case studied in I and II. The results check exactly. This incidentally also provides an additional check on the correctness of the calculations in I and II. The accuracy of the results is therefore firmly established. Since H, G, Φ and Θ are all real for given values of r, m and δ, it then follows that Δc is purely imaginary, i.e. $\Delta c = ic_i$. The stability or instability is determined by whether

$$\left(\frac{G}{H}\right)R_2-\left(\frac{\Phi}{H}\right)\cot\theta \lesseqgtr 0. \tag{18}$$

It may be noted that, whenever a critical Reynolds number exists, the long waves considered here do govern the stability; a result that could be established as in Yih (1963), Lin (1967), and in I, but has not been explicitly proven here.

The main interest in this paper is to consider the physically realistic situation in which the coefficient of dynamic viscosity of the upper fluid is different from that of the lower fluid in addition to the differences in density and depth. The results are indeed quite striking. Figures 4 and 5 show the 'critical' Reynolds

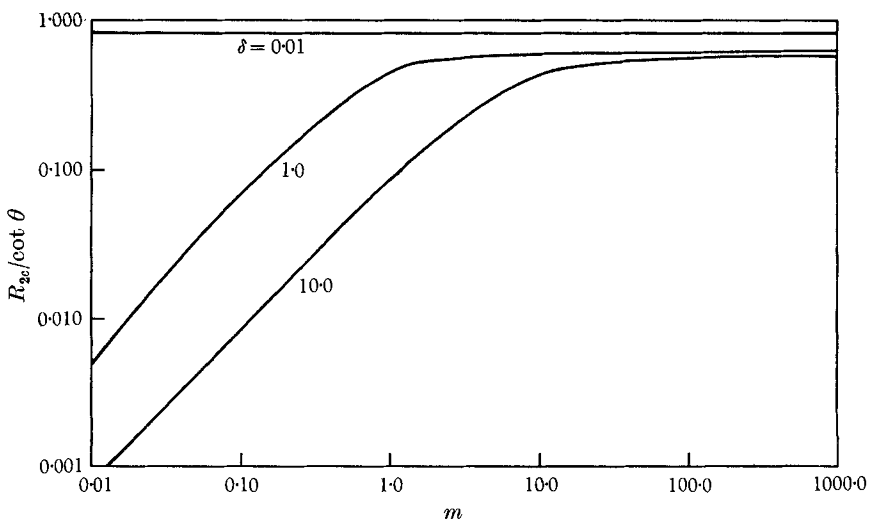

FIGURE 4. 'Critical' Reynolds number $R_{2c}/\cot\theta$ for first mode, with the ratio of density $r = \frac{9}{10}$.

number $R_{2c}/\cot\theta$ for the first and second modes respectively, as a function of m for various values of δ and $r = \frac{9}{10}$. It is readily seen that the second mode is unstable and governs the problem at $m = 1$ but for m smaller than 1 this mode is highly stabilized and the first mode is now the governing mode. The effect of viscosity variation on the first mode is seen to be destabilizing for $m < 1$. For $m > 1$, this effect on both modes is mildly stabilizing. The second mode remains the governing one since it is still unstable at low Reynolds numbers. The overall effect compared with $m = 1$ is thus stabilizing. Figures 6 and 7 are for $r = \frac{1}{10}$.

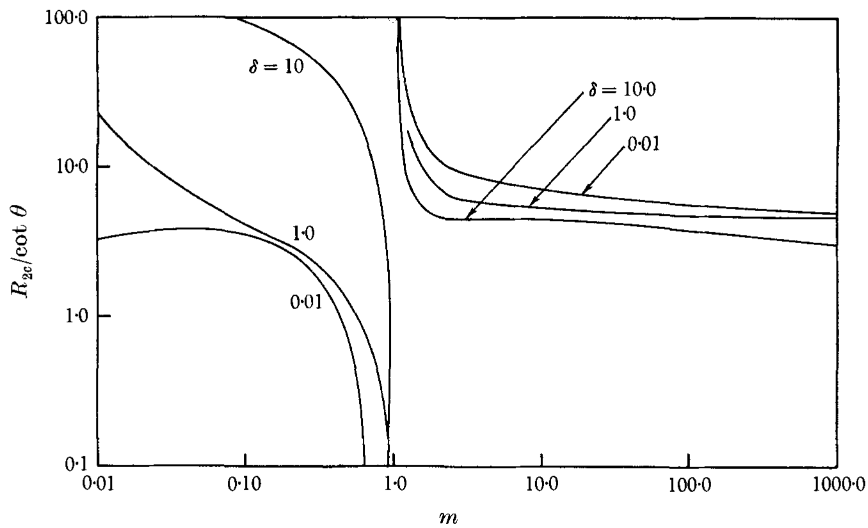

FIGURE 5. 'Critical' Reynolds number $R_{2c}/\cot\theta$ for second mode, with the ratio of density $r = \frac{9}{10}$. The region between the two branches of the stability curve for each fixed δ is the region of instability.

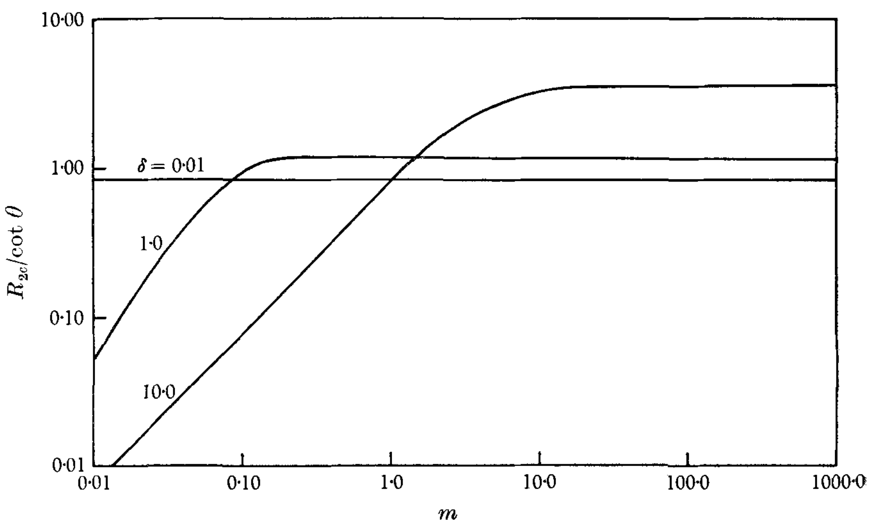

FIGURE 6. 'Critical' Reynolds number $R_{2c}/\cot\theta$ for first mode, with the ratio of density $r = \frac{1}{10}$.

For $\delta > 1{\cdot}6$ the results are similar to the above case, which is typical for all r greater than about $\frac{3}{10}$. However for $\delta < 1{\cdot}6$ the result is not as dramatic, since the first mode always governs even for $m = 1$. Nevertheless, the trends are still similar.

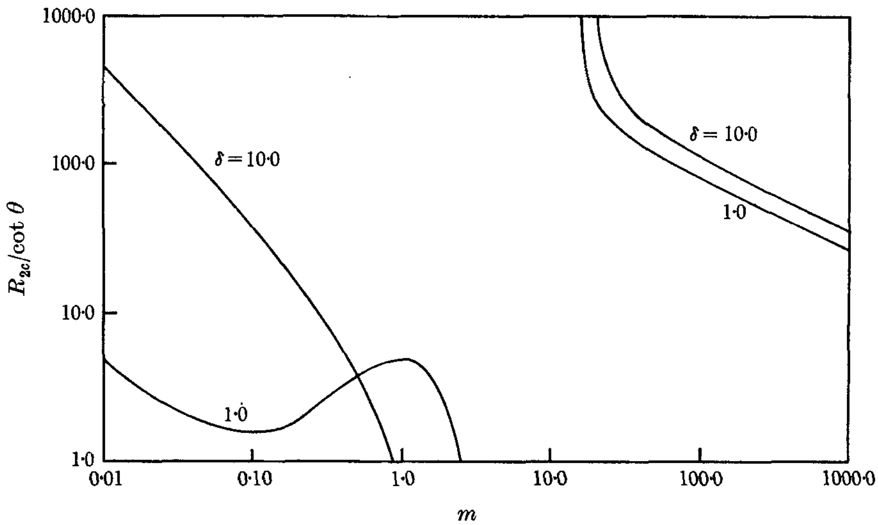

FIGURE 7. 'Critical' Reynolds number $R_{2c}/\cot\theta$ for second mode with the ratio of density $r = \frac{1}{10}$. The region between the two branches of the stability curve for each fixed δ is the region of instability.

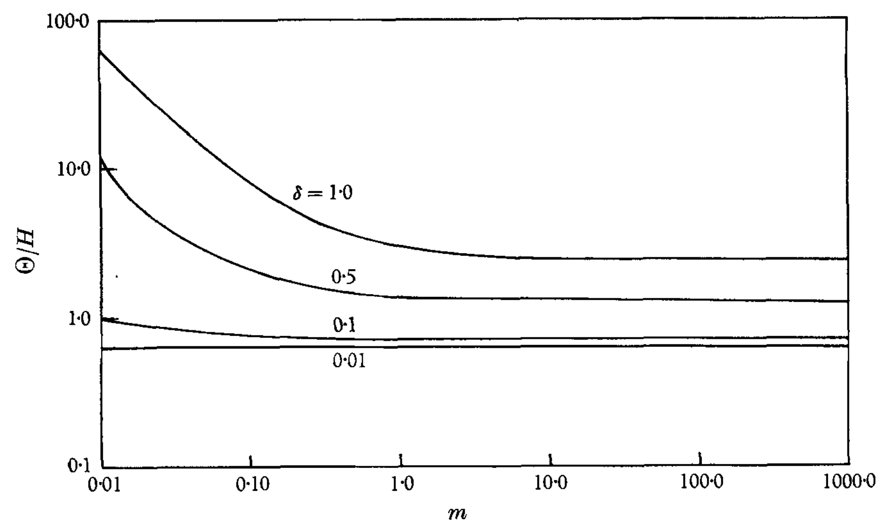

FIGURE 8. Surface tension factor for first mode with $r = \frac{9}{10}$, $q = 1$.

For the sake of completeness the surface tension factor is shown in figures 8 and 9. The effect is stabilizing on the first mode but can be destabilizing on the second mode for $\delta > 1$ and $m > 1$. The surface tension effect is however usually unimportant for the long waves we are considering.

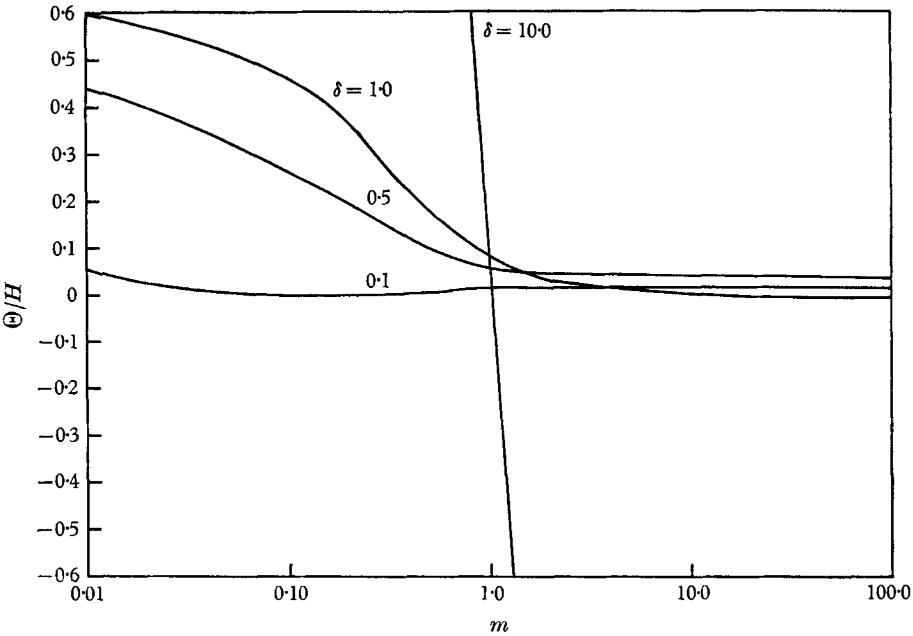

FIGURE 9. Surface tension factor for second mode with $r = \frac{9}{10}$, $q = 1$.

4. The relative stability index

For $m < 1$, it is interesting and relevant to compare the stability of the two-layered flow relative to that of a homogeneous fluid of the same depth. To this end we define, as in I, a relative stability index, S, as follows:

$$S = \frac{\text{critical depth for two-layer flow for a given } \theta}{\text{critical depth for homogeneous flow for same } \theta}.$$

If $S < 1$, the two-layer flow is more unstable than the homogeneous flow. Indeed, if a flow of a homogeneous fluid of depth h is critical, then, when $S < 1$, the replacement of the homogeneous fluid by one with two layers of the same total depth will render the flow unstable. If $S > 1$, the situation is reversed. It may be remarked that the result for the homogeneous case was given by Benjamin (1957) and Yih (1963) and can be obtained from the present treatment as a special case by setting $\delta = 0$ and $S_2 = 0$.

From the definition of the Reynolds number R_2, the critical depth for two-layered flow is given by

$$\{KR_{2c}\mu_2^2(1+\delta)^3/\rho_2^2 g \sin\theta\}^{\frac{1}{3}}$$

and the critical depth for a homogeneous flow is

$$\{3R_c\mu_2^2/\rho_2^2 g \sin\theta\}^{\frac{1}{3}}.$$

Therefore
$$S = (1+\delta)\{KR_{2c}/3R_c\}^{\frac{1}{3}}.$$

Now $R_c = \frac{5}{6}\cot\theta$; therefore

$$S = 0{\cdot}737(1+\delta)(KR_{2c}/\cot\theta)^{\frac{1}{3}}. \tag{19}$$

We have already seen that the first mode is the dominant mode for $m < 1$. The stability index S for the first mode for $m \leqslant 1$ and $r = 0{\cdot}9$ is shown in figure 10. It is seen that $S < 1$, and this is true if r is not too small. (For $r = 0{\cdot}1$, it is possible for $S > 1$.) Since, for $m > 1$, the system is always unstable, it is thus concluded that the presence of the upper layer is generally destabilizing compared with that of a homogeneous fluid of the same total depth.

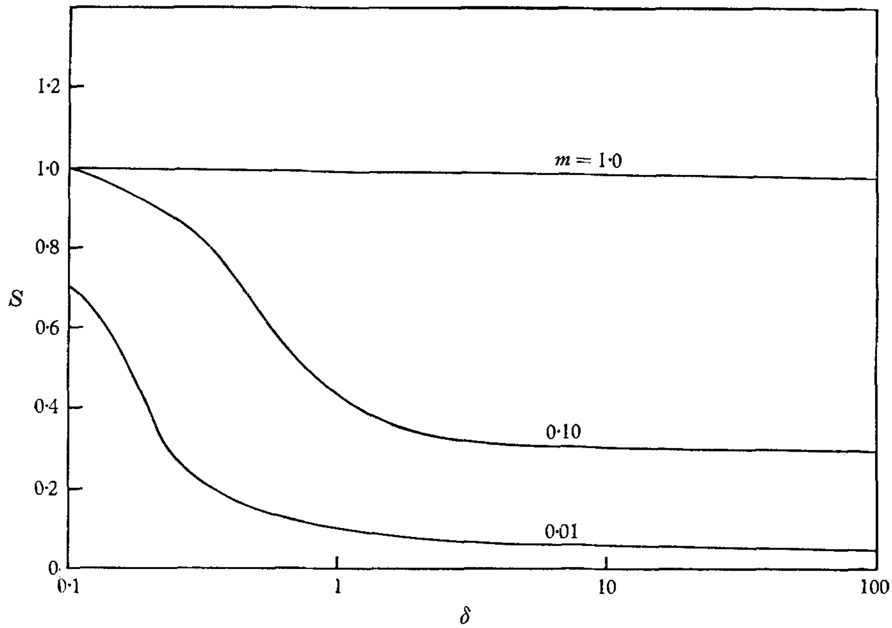

FIGURE 10. Relative stability index for first mode for $m \leqslant 1$, with $r = 0{\cdot}9$.

Finally it may be remarked that an interesting case arises when the upper layer is very thin so that it becomes essentially an adsorbed film and only one degree of freedom is allowed at the surface by definition. Then there can be only one mode that governs the stability of the problem. The situation is analogous to a system of two vibrating masses with one mass shrinking to zero. The second mode is thus ruled out. It is then seen that $S \to 1$ as $\delta \to 0$, so that the present case shows no marked effect on the stability of the layer, in contrast to an adsorbed film of surfactant material exhibiting surface elasticity.

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